Recitation 4: Dataflow Analysis Complexity and Correctness

The termination of the worklist algorithm for dataflow analysis relies on two conditions: the dataflow lattice having finite height and the flow functions being monotonic. A flow function f is monotonic iff $\sigma_1 \sqsubseteq \sigma_2$ implies $f(\sigma_1) \sqsubseteq f(\sigma_2)$ for all σ_1, σ_2.

The correctness of a dataflow analysis depends on the local soundness of its flow functions. For every program configuration c_i in the trace of a program P, a flow function f is locally sound iff $(P \vdash c_i \rightsquigarrow c_{i+1}$ and $\alpha(c_i) \sqsubseteq \sigma_{n_i}$ and $f[\![P[n_i]]\!](\sigma_{n_i}) = \sigma_{n_{i+1}}) \Rightarrow \alpha(c_{i+1}) \sqsubseteq \sigma_{n_{i+1}}$.

Exercises

These exercises prove properties of parity analysis. Assume the following:

- A lattice $(L, \sqsubseteq)$ where $L = \{\top, O, V, \bot\}$ and $\bot \sqsubseteq \{O, V\} \sqsubseteq \top$, $O \sqcup V = \top$
- An abstraction function $\alpha : \mathbb{Z} \mapsto L$, defined as follows:

$$\alpha(n) = \begin{cases} V \text{ when } n \text{ is an even integer } (n \in \{2k : k \in \mathbb{Z}\}) \\ O \text{ when } n \text{ is an odd integer } (n \in \{2k+1 : k \in \mathbb{Z}\}) \end{cases}$$

- a flow function f_P
- initial dataflow analysis assumptions σ_0, in this case σ_0 maps all variables' initial states to $\top$.

1. Disprove the local soundness of the incorrect flow function $f_P[\![a := b]\!](\sigma) = \sigma[a \mapsto O]$

   Proof: *Case* $f_P[\![a := 2]\!](\sigma_{n_i})$.

   Assume $c_i = E_i, n_i$ and $\alpha(E_i) \sqsubseteq \sigma_{n_i}$

   $\sigma_{n_{i+1}} = f_P[\![a := 2]\!](\sigma_{n_i}) = \sigma_{n_i}[a \mapsto O]$ (by definition)

   $c_{i+1} = E_i[a \mapsto 2], n_i + 1$ ($n_{i+1} = n_i + 1$ by rule *step-assign*)

   $\alpha(c_{i+1}) = \alpha(E_i[a \mapsto 2]) = \alpha(E_i)[a \mapsto \alpha_s(2)] = \alpha(E_i)[a \mapsto V]$ (by definition of α and α_s.)

   Notice that $\alpha(E_i) \sqsubseteq \sigma_{n_i}$ by assumption, but $\alpha(c_{i+1}) = \alpha(E_i)[a \mapsto V] \not\sqsubseteq \sigma_{n_i}[a \mapsto O]$ because $\sqsubseteq$ is defined piecewise and $V \not\sqsubseteq_s O$. Therefore $\alpha(c_i) \sqsubseteq \sigma_{n_i} \wedge f_P[\![P[n_i]]\!](\sigma_{n_i}) = \sigma_{n_{i+1}} \not\Rightarrow \alpha(c_{i+1}) \sqsubseteq \sigma_{n_{i+1}}$ $\square$

For the next questions, use the following (correct) flow function for parity analysis:

$$f_P[\![a := b * c]\!](\sigma) = \begin{cases} \sigma[a \mapsto \bot] & \text{if } \sigma(b) = \bot \vee \sigma(c) = \bot \\ \sigma[a \mapsto O] & \text{if } \sigma(b) = O \wedge \sigma(c) = O \\ \sigma[a \mapsto V] & \text{if } (\sigma(b) = V \wedge \sigma(c) \neq \bot) \vee (\sigma(b) \neq \bot \wedge \sigma(c) = V) \\ \sigma[a \mapsto \top] & \text{if } (\sigma(b) = \top \wedge \sigma(c) \notin \{V, \bot\}) \vee (\sigma(b) \notin \{V, \bot\} \wedge \sigma(c) = \top) \end{cases}$$

2. Prove the monotonicity of $f_P[\![a := b * c]\!](\sigma)$ for the case $(\sigma(b) = V \wedge \sigma(c) \neq \bot) \vee (\sigma(b) \neq \bot \wedge \sigma(c) = V)$

   Proof of monotonicity of the above flow function

   Assume $\sigma_1 \sqsubseteq \sigma_2$

   $\sigma_1(b) \sqsubseteq_s \sigma_2(b)$ and $\sigma_1(c) \sqsubseteq_s \sigma_2(c)$ (Since $\sqsubseteq$ is defined point-wise)

   Case $(\sigma_1(b) = V \wedge \sigma_1(c) \neq \bot) \vee (\sigma_1(b) \neq \bot \wedge \sigma_1(c) = V)$

   Since $\sigma_1(b) \sqsubseteq_s \sigma_2(b)$ and $\sigma_1(c) \sqsubseteq_s \sigma_2(c)$:

   $(\sigma_2(b) \in \{V, \top\} \wedge \sigma_2(c) \neq \bot) \vee (\sigma_2(c) \in \{V, \top\} \wedge \sigma_2(b) \neq \bot)$

   $$\therefore f_P[\![a := b * c]\!](\sigma_2) = \begin{cases} \sigma_2[a \mapsto V] & \text{if } (\sigma_2(b) = V \wedge \sigma_2(c) \neq \bot) \vee \\ & \quad (\sigma_2(c) = V \wedge \sigma_2(b) \neq \bot) \\ \sigma_2[a \mapsto \top] & \text{otherwise} \end{cases}$$

   Since $\sqsubseteq$ is defined point-wise, $V \sqsubseteq_s V$, $V \sqsubseteq_s \top$, and $\sigma_1 \sqsubseteq \sigma_2$, we get

   $f_P[\![a := b * c]\!](\sigma_1) = \sigma_1[a \mapsto V] \sqsubseteq f_P[\![a := b * c]\!](\sigma_2)$

   $\square$

3. Prove the local soundness of $f_P[\![a := b * c]\!](\sigma)$ Now let's try showing that the above function is locally sound. Remember, a flow function f is locally sound iff $(P \vdash c_i \rightsquigarrow c_{i+1}$ and $\alpha(c_i) \sqsubseteq \sigma_{n_i}$ and $f[\![P[n_i]]\!](\sigma_{n_i}) = \sigma_{n_{i+1}}) \Rightarrow \alpha(c_{i+1}) \sqsubseteq \sigma_{n_{i+1}}$

   Proof of local soundness of the above flow function:

   Assume $f_P\ c_i = E_i, n_i$ and $\alpha(E_i) \sqsubseteq \sigma_{n_i}$

   Then $c_{i+1} = E_i[a \mapsto m], n_i + 1$ for some m such that $E_i(b) * E_i(c) = m$ since $n_{i+1} = n_i + 1$ by rule *step-arith*

   Now $\alpha(c_{i+1}) = \alpha(E_i[a \mapsto m]) = \alpha(E_i)[a \mapsto \alpha_s(m)]$ by the definitions of α and α_s

Case $m \in \{2k : k \in \mathbb{Z}\}$

Then $\alpha_s(m) = V$ and $E_i(b)$ is even or $E_i(c)$ is even

Thus $(\alpha_s(E_i(b)) = V \wedge \alpha_s(E_i(c)) \neq \bot) \vee$
$(\alpha_s(E_i(c)) = V \wedge \alpha_s(E_i(b)) \neq \bot)$

Since $\alpha(E_i) \sqsubseteq \sigma_{n_i}$, we get
$(\alpha_s(E_i(b)) = V \sqsubseteq_s \sigma_{n_i}(b) \wedge \alpha_s(E_i(c)) \neq \bot \sqsubseteq_s \sigma_{n_i}(c)) \vee$
$(\alpha_s(E_i(c)) = V \sqsubseteq_s \sigma_{n_i}(c) \wedge \alpha_s(E_i(b)) \neq \bot \sqsubseteq_s \sigma_{n_i}(b))$

From this we get $(\sigma_{n_i}(b) \in \{V, \top\} \wedge \sigma_{n_i}(c) \neq \bot) \vee (\sigma_{n_i}(c) \in \{V, \top\} \wedge \sigma_{n_i}(b) \neq \bot)$

$$\therefore \sigma_{n_{i+1}} = f_P[\![a := b * c]\!](\sigma_{n_i}) = \begin{cases} \sigma_{n_i}[a \mapsto V] & \text{if } (\sigma_{n_i}(b) = V \wedge \sigma_{n_i}(c) \neq \bot) \vee \\ & \quad (\sigma_{n_i}(c) = V \wedge \sigma_{n_i}(b) \neq \bot) \\ \sigma_{n_i}[a \mapsto \top] & \text{otherwise} \end{cases}$$

Since $\sqsubseteq$ is defined point-wise, $\alpha(E_i) \sqsubseteq \sigma_{n_i}$, $V \sqsubseteq_s V$, and $V \sqsubseteq_s \top$, we get
$\alpha(c_{i+1}) = \alpha(E_i)[a \mapsto V] \sqsubseteq \sigma_{n_{i+1}}$

Case $m \in \{2k + 1 : k \in \mathbb{Z}\}$

[Similar to the previous case and left up to the reader to prove as an exercise]

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